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NONSTATIONARY NONLINEAR HEAT-CONDUCTION PROBLEMS

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An exact analytical solution is constructed for the one-dimensional nonlinear nonstationary problem of heat conductivity with boundary conditions of the third kind.

The solution of the nonstationary one-dimensional problem of heat conductivity in the absence of heat sources and sinks within the body results in the need to investigate the equation [1]

$$\rho c\frac{\partial T}{\partial t}=\frac{\partial\mu}{\partial r}\cdot\frac{\partial T}{\partial r}+\frac{\nu\mu}{r}\cdot\frac{\partial T}{\partial r}+\mu\frac{\partial^2 T}{\partial r^2}. \tag{1}$$

Let us find the temperature T = T(r, t) satisfying this equation, the initial condition

$$T(r,0)=\varphi(r) \tag{2}$$

and the boundary conditions of the third kind

$$\left(\mu\frac{\partial T}{\partial r}+\mu_1 T\right)_{r=l}=0;\quad \left(\mu\frac{\partial T}{\partial r}-\mu_2 T\right)_{r=-l}=0, \tag{3}$$

where the heat-exchange coefficients $\mu_1 = \mu_1(r, t, T)$ and $\mu_2 = \mu_2(r, t, T)$ are given or empirical parameters.

Limiting ourselves to an examination of the symmetric problem, let us take [1]

$$\left(\frac{\partial T}{\partial r}\right)_{r=0}=0. \tag{4}$$

in place of the second condition in (3). The values $\nu = 0, 1, 2$ determine the plane, cylindrical, or spherical symmetry of the body, respectively.

Let there be the dependences

$$\rho c=P(t)F(r)T^p;\quad \mu=Q(t)G(r)T^q. \tag{5}$$

Then (1) is reduced to the form

$$\frac{T^{p-q+1}}{\lambda(t)}\cdot\frac{\partial T}{\partial t}=\frac{G(r)}{F(r)}\left\{T\frac{\partial^2 T}{\partial r^2}+\left[\frac{G'(r)}{F(r)}+\frac{\nu}{r}\right]T\frac{\partial T}{\partial r}+q\left(\frac{\partial T}{\partial r}\right)^2\right\}, \tag{6}$$

where $\lambda(t) = Q(t)/P(t)$.

We construct the solution of (6) in the form of the product

$$T=\tau(t)R(r). \tag{7}$$

Inserting (7) into (6) and separating variables, we obtain two equalities

$$\tau^{p-q-1}\frac{d\tau}{dt}=-k^2\lambda(t), \tag{8}$$

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$$RR'' + qR'^2 + f(r)\,RR' + k^2 g(r)\,R^{p-q+2} = 0. \tag{9}$$

We have here introduced the notation

$$f(r) = \frac{d}{dr}\ln[G(r)\,r^{\nu}];\; g(r) = \frac{F(r)}{G(r)}. \tag{10}$$

If $p \neq q$, then we obtain from (8)

$$\tau = \left(A - (p-q)\,k^2 \int \lambda(t)\,dt\right)^{1/(p-q)}. \tag{11}$$

For $p = q$ we find in place of (11)

$$\tau = B\exp\left(-k^2 \int \lambda(t)\,dt\right). \tag{12}$$

The solution for even (9) can be written for certain values of p and q. For instance, for $p = q = -1$, it it will have the form

$$R = C_1 \exp\left\{\int \frac{1}{G(r)\,r^{\nu}}\left[C_2 - k_2 \int r^{\nu} F(r)\,dr\right]dr\right\}. \tag{13}$$

If $p = q \neq -1$, then performing the substitution $R = u^{1/(1+q)}$ in (9) and giving the functions F(r) and G(r) in the form

$$F(r) = D/(r^{\nu}\sqrt{1+r^2});\; G(r) = (D\sqrt{1+r^2})/r^{\nu},$$

we arrive at the equation

$$(1+r^2)\,u'' + ru' + (1+q)\,k^2 u = 0.$$

By using the substitution $u = \eta(\xi)$ ($\xi = \operatorname{arsh} r$) it can be reduced to the form

$$\eta'' + (1+q)\,k^2\eta = 0.$$

The solution of this equation is

$$\eta(r) = a\sin\left(k\sqrt{1+q}\,\operatorname{arsh} r + C_2\right).$$

Finally, the solution of (9) is now governed by the function

$$R = C_1\left[\sin\left(k\sqrt{1+q}\,\operatorname{arsh} r + C_2\right]^{1/(1+q)}.$$

Let us examine the case when $F = G = 1$, $\nu = 0$, $p-q = 1$. Then (9) is simplified

$$RR'' + qR'^2 + k^2R^3 = 0. \tag{14}$$

Making the substitution

$$R' = R^{3/2}u(\ln R), \tag{15}$$

we arrive at the equation

$$uu' + (q + 3/2)\,u^2 + k^2 = 0.$$

We hence find

$$\ln R = -1/(2q+3)\ln[(q+3/2)\,u^2 + k^2] + \text{const}.$$

or

$$u = \left(k\sqrt{C_1^2 - R^{2q+3}}\right)/\left(\sqrt{q+3/2}\;R^{q+3/2}\right). \tag{16}$$

From (15) and (16) there follows

$$x + C_2 = \left(\sqrt{q+3/2}\right)/\,k\int\frac{R^q\,dR}{\sqrt{C_1^2 - R^{2q+3}}}\quad (r = x). \tag{17}$$

The substitution

$$C_1^2 - R^{2q+3} = z^2 \tag{18}$$

results in the quadrature

$$x + C_2 = -1/\left(k\sqrt{q+3/2}\right)\int \frac{dz}{(C_1^2 - z^2)^s}, \tag{19}$$

where $s = (q + 2)/(2q + 3)$.

Selection of the value of q ($q \neq -3/2$) determines some value of the quadrature (19). Inserting the expression found for z from (19) into (18), we arrive at the solution of (14).

By using (7), (12), and (13) we satisfy the boundary condition (3) for the symmetric problem by considering here that

$$F(r) = r^m;\ G(r) = r^n;\ \lambda = \lambda_0 = \text{const}.$$

We then obtain in place of (7)

$$T = A\exp\left\{-k^2\lambda_0 t + \frac{cr^{1-\nu-n}}{1-\nu-n} - \frac{k^2 r^{2+m-n}}{(\nu+m+1)(m-n+2)}\right\}. \tag{20}$$

If we set $m - n + 1 > 0$, then condition (4) will evidently be satisfied for $c = 0$. Moreover, assuming that

$$\mu_1 = \mu_2 = \mu_0 T^{-1} Q(t)$$

and satisfying the first condition in (3) by using (20), we determine the constant k^2 of the separation of variables

$$k^2 = \mu_0(\nu + m + 1)/(l^{1+m}).$$

The initial condition $\varphi(r)$ can hence not be arbitrary but should have the form

$$\varphi(r) = B\exp\left[-\frac{\mu_0 r^{2+m-n}}{(m-n+2)\,l^{1+m}}\right].$$

The constants B, μ_0, m, n entering here are determined by the singularities of the problem.

NOTATION

T	is the temperature;
r, x	are the linear coordinates;
t	is the time;
ρ	is the specific gravity;
c	is the specific heat;
μ_1, μ_2	are the coefficient of heat conductivity;
T(r, t), τ(t), R(r)	are the coefficients of heat exchange;
φ(r), P(t), Q(t), F(r), G(r)	are the desired functions;
ν, p, q, l, m, n, D, λ_0, μ_0	are the given functions;
A, B, C_1, C_2, k, a	are the given constants;
A, B, C_1, C2, k, a	are the arbitrary constants.

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